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Superspin classification for supersymmetry representations with unrestricted central charges

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Abstract. A generalised superspin Casimir operator for the global supersymmetry algebra with arbitrary central charges is constructed. It is noted that the spin-reducing central charge is naturally singled out as a special case, and the special conditions necessary for this case are extracted from the general Casimir.

1. Introduction

In an earlier paper (Rands and Taylor 1983) the analysis of off-shell supersymmetry (SUSY) representations (Bufton and Taylor 1983, Ferrara and Savoy 1982, Ferrara *et al* 1981, Nahm 1978, Pickup and Taylor 1981, Rittenberg and Sokatchev 1981, Taylor 1982) was extended to the situation where central charges are present in the SUSY algebra. However, the central charges considered there were restricted by the assumption that the product of the central charge and its complex conjugate was proportional to the delta function:

$$Z^{*ij}Z^{jk} = \kappa p^2 \delta_i^j \tag{1}$$

(where κ is a constant). This includes the analysis (Galperin *et al* 1983) of USp(N) SUSY with one central charge, where equation (1) automatically holds.

Here we show that this assumption can, in fact, be relaxed and that the superspin classification extends straightforwardly to cases where $Z^{*ij}Z^{jk}$ is an arbitrary operator. Having done this it is seen that the degenerate, spin-reducing case ($\kappa = 1$ in equation (1)) is naturally singled out, and that the conditions on the SUSY generators necessary to accommodate this case coincide with those obtained in the restricted analysis of Rands and Taylor (1983).

2. Construction of the superspin Casimir

Following the notation of Rands and Taylor (1983), we choose the N-extended SUSY algebra, $\mathscr{S}_N$, to have

chiral SUSY generators	$S_{\alpha +}{}^{l}(1 \leqslant l \leqslant N)$
their complex conjugates	$S_{\alpha - l}$
antisymmetric complex central charge generators	Z^{ij}
their complex conjugates	Z^{*ij}

together with Poincaré generators P_μ, $J_{\mu\nu}$ and appropriate internal symmetry generators.

Z^{ij}, Z^{*ij} commute with $S_{\alpha+}{}^l$, $S_{\alpha-l}$, P_μ and $J_{\mu\nu}$, and among themselves. We also have the following (anti-) commutation relations, together with their conjugates:

$$[S_{\alpha+}{}^l, S_{\beta-m}]_+ = -2(\not{p}C)_{\alpha+\beta-}\delta^l{}_m \tag{2a}$$

$$[S_{\alpha+}{}^l, S_{\rho+}{}^m]_+ = 2C_{\alpha+\beta+}Z^{lm} \tag{2b}$$

$$[S_{\alpha-l}, S_{\beta-m}]_+ = 2C_{\alpha-\beta-}Z^{*lm} \tag{2c}$$

$$[J_{\mu\nu}, S_{\alpha+}{}^l]_- = \mathrm{i}(\sigma_{\mu\nu}S_+{}^l)_{\alpha+} \tag{2d}$$

$$[P_\mu, S_{\alpha+}{}^l]_- = 0 \tag{2e}$$

where C is the charge conjugation matrix, $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$ and we take the metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The superspin Casimir is constructed by modifying the Pauli–Lubanski vector $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}J^{\mu\nu}p^\sigma$ by addition of further terms to obtain a real conserved vector C_μ which commutes with $S_{\alpha+}{}^l$.

To do this, we consider

$$K_\mu = \bar{S}_+{}^l\gamma_{\mu'}S_{-m}f_l^m - \bar{S}_{-l}\gamma_{\mu'}S_+{}^m f_m{}^l \tag{3a}$$

(where $\gamma_{\mu'} = \gamma_\mu - p_m\not{p}/p^2$, so that $p^\mu\gamma_{\mu'} = 0$) and

$$J_\mu = p^\nu[\bar{S}_+{}^l\sigma_{\mu\nu}S_+{}^m g_{lm} - \bar{S}_{-l}\sigma_{\mu\nu}S_{-m}g^{lm}] \tag{3b}$$

with $f_m{}^l$, g_{lm} and g^{lm} functions of Z^{ij} and Z^{*ij} and restricted by the requirement of the reality of K_μ and J_μ:

$$(f_l^m)^* = f_m{}^l \qquad (g_{lm})^* = g^{lm} \qquad g_{ml} = -g_{lm}. \tag{4a, b, c}$$

Then, the general linear combination

$$C_\mu = W_\mu - \alpha K_\mu - \beta p^{-2}J_\mu \qquad \alpha, \beta \text{ real} \tag{5}$$

after some algebra has the following commutation relation with $S_{\alpha+}{}^l$:

$$[C_\mu, S_{\alpha+}{}^l]_- = \tfrac{1}{2}(\gamma_{\mu'}\not{p}S_+{}^l)_{\alpha+}\{\delta_l^i - 8\alpha f_l^i - 4\beta p^{-2}Z^{im}g_{lm}\}$$
$$+ (4\alpha Z^{il}f_l^m - 2\beta g^{im})(\gamma_{\mu'}S_{-m})_{\alpha+}. \tag{6}$$

For this to vanish, we require first

$$\beta g^{im} = 2\alpha Z^{il}f_l^m \tag{7a}$$

so that

$$\beta g_{im} = 2\alpha Z^{*il}f_m{}^l \qquad \text{(by conjugation)} \tag{7b}$$

and secondly

$$\delta_l^i - 8\alpha f_l^i - 4p^{-2}Z^{im}2\alpha Z^{*ls}f_m{}^s$$

i.e.

$$\delta_l^i = 8\alpha[f_l^i - p^{-2}Z^{im}f_m{}^sZ^{*sl}]. \tag{8}$$

This equation may be solved by taking

$$f_l^i = (f_i^l)^* = (1/8\alpha)\{\delta_l^i + p^{-2}Z^{*lj}Z^{ji} + p^{-4}Z^{*lj}Z^{jk}Z^{*km}Z^{mi} + \ldots\}$$
$$= (1/8\alpha)[(1-p^{-2}Z^*Z)^{-1}]_l^i. \tag{9}$$

Then

$$\beta g_{im} = \tfrac{1}{4}Z^{*il}f^{(0)}{}_m{}^l = \tfrac{1}{4}f^{(0)}{}_i{}^k Z^{*km} \tag{10a}$$
$$\beta g^{im} = \tfrac{1}{4}Z^{il}f^{(0)}{}_l{}^m = \tfrac{1}{4}f^{(0)}{}_k{}^i Z^{km} \tag{10b}$$

where

$$f^{(0)}{}_l{}^m = (f^{(0)}{}_m{}^l)^* = (\delta_l^m + p^{-2}Z^{*li}Z^{im} + p^{-4}Z^{*li}Z^{ij}Z^{*jk}Z^{kn} + \ldots) = 8\alpha f_l^m$$
$$= [(1-p^{-2}Z^*Z)^{-1}]_l^m. \tag{11}$$

Thus W_μ can be extended to a Casimir by taking

$$C_\mu = W_\mu - \tfrac{1}{8}K'_\mu - \tfrac{1}{4}J'_\mu \tag{12}$$

for

$$K'_\mu = \bar{S}_+{}^l\gamma_\mu S_{-m}f^{(0)}{}_l{}^m - \bar{S}_{-l}\gamma_\mu S_+{}^m f^{(0)}{}_m{}^l \tag{13a}$$
$$J'_\mu = (p^\nu/p^2)(\bar{S}_+{}^l\sigma_{\mu\nu}S_+{}^m Z^{*ln}f^{(0)}{}_m{}^n - \bar{S}_{-l}\sigma_{\mu\nu}S_{-m}Z^{ln}f^{(0)}{}_n{}^m). \tag{13b}$$

This will give rise to a degenerate case when $(1-p^{-2}Z^*Z)^{-1}$ does not exist, i.e. when

$$Z^{*il}Z^{lm} = p^2\delta_i^m. \tag{14}$$

We will consider this case in a little more detail shortly. For the non-degenerate cases, we have $C^2 = C_\mu C^\mu$ as a Casimir of the algebra $(S_{\alpha+}{}^l, S_{\alpha-l}, P_\mu, J_{\mu\nu})$, and further algebra shows that in the rest frame, $P_\mu = (m, \mathbf{0})$, Ci satisfies the SU(2) algebra, and so C^2 takes values $p^2Y(Y+1)$ for $Y = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. For $Z^{ij} = 0$, C_μ reduces to the value given by Taylor (1982). For values of Z^{ij} such that equation (1) holds ($\kappa \neq 1$), C_μ reduces to the value given by Rands and Taylor (1983).

We thus have a suitable generalisation of superspin to the general (non-degenerate) central charge situation (modulo internal symmetries) with an identical superspin spectrum to the non-central charge case.

3. The degenerate case

We have seen that for the case where $Z^{*ij}Z^{jk} = p^2\delta_i^k$, the value of C_μ appears to become singular. To deal with this, we rewrite the offending part

$$K'_\mu + 2J'_\mu = \bar{S}_+{}^l\gamma_\mu S_{-m}f^{(0)}{}_l{}^m - \bar{S}_{-l}\gamma_\mu S_+{}^m f^{(0)}{}_m{}^l + p^{-2}\bar{S}_+{}^l\gamma_\mu \not{p} S_+{}^n f^{(0)}{}_l{}^m Z^{*mn}$$
$$- p^{-2}\bar{S}_{-l}\gamma_\mu \not{p} S_{-n}f^{(0)}{}_m{}^l Z^{mn}$$

(since $p^\nu\sigma_{\mu\nu} = \frac{1}{2}\gamma_\mu \not{p}$, and equations (10) apply)

$$= \bar{S}_+{}^l\gamma_\mu[S_{-m} + p^{-2}\not{p}S_+{}^n Z^{*mn}]f^{(0)}{}_l{}^m - \bar{S}_{-l}\gamma_\mu[S_+{}^m + p^{-2}\not{p}S_{-n}Z^{mn}]f^{(0)}{}_m{}^l. \tag{15}$$

For this to have a finite limit as $Z^{*ij}Z^{jk} \to p^2\delta_i^k$, each of the terms in the square brackets must vanish, i.e.

$$S_{-m} = -p^{-2}\not{p}S_+{}^n Z^{*mn} \qquad S_+{}^m = -p^{-2}\not{p}S_{-n}Z^{mn}$$

or

$$\not{p}S_+{}^n = Z^{mn}S_{-m} \qquad \not{p}S_{-n} = Z^{*mn}S_+{}^m. \tag{16}$$

These two conditions (one is actually the complex conjugate of the other) are identical to those obtained by Rands and Taylor (1983) and we thus see that the degenerate case arises in exactly the same way as before.

The superspin Casimir then reduces to the value in (Rands and Taylor 1983), i.e.

$$C_\mu = W_\mu + \tfrac{1}{8}p^{-2}\bar{S}_+{}^l\gamma_\mu\not{p}S_+{}^m Z^{*lm} \tag{17}$$

and discussion of this case can then proceed in exactly the same way.

4. Comment

It is unfortunate that an analysis of internal symmetry along the lines of Rands and Taylor (1983) can no longer be conducted, since the pseudo-symplectic metric used there is not present in the case of the unrestricted central charges. However, we can see, by considering how $S_{\alpha+}{}^l$, $S_{\alpha-l}$, Z^{ij} and Z^{*ij} will transform under an internal symmetry transformation, that C_μ, and therefore C^2, will remain invariant under such a transformation.

Thus we have a satisfactory superspin classification for all supersymmetry representations without any restrictions being applied to the type of central charge allowed. This should allow for a more satisfactory use of integration over the space of central charge coordinates, z_{ij}, $z^*{}_{ij}$, in the construction of superspace Lagrangians (Restuccia and Taylor 1983).

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